Induction

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1 The Induction Mindset

Many problems have a relatively simple solution with induction but a much more complicated solution without induction. It is worthwhile always at least trying to induct if you are stuck on a problem. Induction can give free conditions if you have nothing to work with and thinking recursively can yield nontrivial observations. Here some views on induction:

1. *Free conditions:* See what cases you can reduce to by assuming the result up to $n-1$. This is often useful when there is very little to work with and it is hard to get started in a problem.

2. *Greedy algorithms:* A common approach in induction is to remove some extremal element and produce a modified subproblem. Applying the induction hypothesis along with properties of the removal procedure then yields the result. This approach generally corresponds to greedy algorithms and typically requires at most n recursive calls to the induction hypothesis. These proofs can usually be written up without induction by expanding the steps of the greedy algorithm.

3. *Real induction:* A term coined by Jacob – this refers to cases where the induction hypothesis is applied many times to different subproblems. These induction proofs tend to have the flavor of exhausting cases. The resulting algorithms are roughly analogous to dynamic programming and require more than n recursive calls to the induction hypothesis. Note that unlike devising dynamic programming algorithms, your induction proofs do not need to be efficient! Unlike greedy algorithms, real induction proofs often cannot be expanded easily into non-recursive algorithms. As a result, these proofs do not usually have a non-inductive analogue (hence the name real induction).

4. *Induction that is not just removal:* Sometimes removing an element is not sufficient but induction can still be applied to a smaller problem. These proofs tend to be harder, so watch for sneaky ways to find a smaller problem!

5. *Strong induction:* Remember you can always assume the result is true for all $m < n$ and not just $n-1$.

6. *Strengthening the problem:* Sometimes the induction hypothesis is just not strong enough to make the induction work but a stronger hypothesis is.

2 Examples

We start with two example problems which illustrate various ways to think about induction.

Example 1. *(USAMO 2008) Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.*

Proof. Set $k_0 = 3$, which satisfies the condition. Assume the result is true up to n with $k_0 k_1 \cdots k_n - 1 = a(a+1)$. Setting $k_{n+1} = a^2 - a + 1$ now yields that

$$k_0 k_1 \cdots k_{n+1} - 1 = (a^2 + a + 1)(a^2 - a + 1) - 1 = a^2(a^2 + 1)$$

Since $(a+1)k_{n+1} - (a-1)k_0 k_1 \cdots k_n = 2$, it follows that k_{n+1} is coprime to $k_0, k_1, \dots, k_n$ since it is odd, completing the induction. □

This next problem illustrates the usefulness of strengthening a problem to the point where induction starts to work.

Example 2. *(USAMO 2007) Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.*

The key is to generalize the problem to make the decrease the step size of the induction. We instead prove the following result and set $m = n$ to solve the problem.

Lemma 1. *Let S be a set containing $m(n+1) - 1$ elements, for some positive integers m and n. If the n-element subsets of S are partitioned into two classes, then there are m pairwise disjoint sets in the same class.*

Proof. We prove this by induction on m. If $m = 1$, then $|S| = n$ and S itself suffices. Assume the result is true for m and suppose that S satisfies $|S| = (m+1)(n+1) - 1$ and all of its n-element subsets have been labelled red or blue. Let T be any subset of S of size $m(n+1) - 1$. By the induction hypothesis, there are m pairwise disjoint subsets of T of the same color, which we assume is blue. The $n+1$ subsets of size n of the remaining $n+1$ elements must all be red, since otherwise the induction would be complete. Applying this for all such T yields that any two subsets that differ in one element must have the same color. Therefore all n-element subsets are of the same color, completing the induction. □

What is different about these two problems? The second problem is an instance of real induction while the first is not. While induction was essential to motivating the solution to the first problem, we could easily write up the solution without induction. Set

$$k_n = 2^{2^{n+1}} - 2^{2^n} + 1 = \frac{2^{2^{n+2}} + 2^{2^{n+1}} + 1}{2^{2^{n+1}} + 2^{2^n} + 1}$$

and note that telescoping yields the result. In contrast, the solution to the second problem is difficult to write up without induction as we apply the induction hypothesis everywhere.

In the next problem, we greedily remove an element and apply induction. Like the first problem, this solution can also be written up without induction. However, thinking "inductively" makes solving this problem much easier.

Example 3. *There are finitely many congruent, parallel squares in a plane such that among any $k+1$ squares some two intersect. Show that the squares can be divided into at most $2k-1$ nonempty groups such that all squares in the same group have a common point.*

Proof. We prove the claim by induction on k. Consider the square A with minimum y-coordinate and the set S of all squares that intersect A. Note that all squares in S must contain one of the two upper vertices of A. Now consider removing A and all squares in S. Assume for contradiction that there are k remaining squares such that no two intersect. Appending A to these k squares gives $k+1$ that are pairwise disjoint, contradicting the condition in the problem. Thus among any k remaining squares, some two intersect, and by the induction hypothesis there are $2k-3$ points such that all squares not removed pass through at least one of these points. Adding the two upper vertices of A to these points completes the induction.

We now show the base case $k = 1$. Consider the square B with minimal y-coordinate. Every other square passes through an upper vertex of B. If every square passes through one of these vertices, we are done. Otherwise, take the square C of minimum x-coordinate through the upper left vertex of B and the square D of maximum x-coordinate through the upper right vertex of B. These have a common point on B which all squares pass through. □

This last problem we present gives a very nontrivial reduction to a smaller problem and illustrates the fact that sometimes you really need to modify rather than just remove an element.

Example 4. *(USAMO 2008) At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).*

The difficult part of this problem is showing that there is at least one way to split the mathematicians between the two rooms. We will prove this by induction on the number of mathematicians. We leave it as an exercise to you to modify the induction to get the full result. We also note that an argument from linear algebra over $\mathbb{F}_2^n$ shows that as long as there is one way, the number of ways is a power of two. I encourage anyone who knows some linear algebra to verify this for fun.

Lemma 2. *For any graph G on n vertices, there is a way to partition the vertices of G into two sets A and B such that each vertex has even degree within its side of the partition.*

Proof. The lemma trivially holds for $n = 1$. Assume it is true for all graphs on $n-1$ vertices. If every vertex has even degree, then put all of the vertices on one side of the partition. Therefore we may assume there is some vertex v of odd degree. Let S be its set of neighbors. Consider the graph G' formed by removing v and reversing all of the edges between vertices in S. By the induction hypothesis applied to G', there is some partition $A \cup B$ of its $n-1$ vertices with the property. Since $|S|$ is odd, one of A and B has an even number of vertices of S. Suppose this set is A. It follows that since $S \cap B$ is odd, reversing the edges in $S \cap B$ preserves the fact that every vertex in B has even degree. Reversing the edges in $S \cap A$ causes all of the vertices in $S \cap A$ to have odd degree in A. Adding v to A now yields the desired partition, completing the induction. □

3 Problems

Some of these problems are entirely solved by induction while others only use induction in one part of their solution. Feel free to ask me if you would like a hint.

1. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

   (a) the union of any two white subsets is white;

   (b) the union of any two black subsets is black;

   (c) there are exactly N white subsets.

2. Given a set A of positive integers, let $S(A) = \{a + b : a, b \in A, a \neq b\}$. Prove that there are infinitely many positive integers n such that there exist distinct A, B with $|A| = |B| = n$ and $S(A) = S(B)$.

3. Given a finite number of boys and girls, a sociable set of boys is a set of boys such that every girl knows at least one boy in that set; and a sociable set of girls is a set of girls such that every boy knows at least one girl in that set. Prove that the number of sociable sets of boys and the number of sociable sets of girls have the same parity.

4. James draws N lines lie on a plane, no two of which are parallel and no three of which are concurrent. Prove that there exists a non-self-intersecting broken line $A_0A_1A_2A_3...A_N$ with N parts, such that on each of the N lines lies exactly one of the N segments of the line.

5. In an $m \times n$ matrix of distinct positive integers, we color the $p \leq m$ largest numbers in each column white, and the $q \leq n$ largest numbers in each row black. Prove that at least pq numbers have been colored twice.

6. Every road in the country Graphsville is one way, and every pair of cities is connected by exactly one road. Show that there is a city which can be reached by every other city either directly or by going through at most one other city.

7. Fix an integer $n \geq 2$. An $n \times n$ sieve is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A stick is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

8. There are 100 apples on the table with total weight of 10 kg. Each apple weighs no less than 25 grams. The apples need to be cut for 100 children so that each of the children gets 100 grams. Prove that you can do it in such a way that each piece weighs no less than 25 grams.

9. 110 teams participate in a volleyball tournament. Every team has played every other team exactly once (there are no ties in volleyball). Turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

10. You are given $\binom{n}{2}$ stones, divided into piles of various sizes. Each minute, you take one stone from each existing pile, and group them together into a new pile. Prove that eventually, you will have one pile of size i for each $1 \leq i \leq n$.

11. Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, ..., A_k$ of X is tight if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

12. Each square of a $(2^n-1)\times(2^n-1)$ square board contains either $+1$ or -1. Such an arrangement is deemed successful if each number is the product of its neighbours. Find the number of successful arrangements.

13. Let $p > 2017$ be a prime number and let $a_1, a_2, \ldots, a_{p-2}$ be positive integers such that p does not divide a_k or $a_k^k - 1$ for any k. Prove that there are indices $1 \leq s_1 < s_2 < \cdots < s_k \leq p-2$ such that $a_{s_1}a_{s_2}\cdots a_{s_k} - 2017$ is divisible by p.

14. The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been coloured with the following property being satisfied: for each $0 \leq i, j \leq 2^n - 1$, the j-th cell in the i-th row and the $(i+j)$-th cell in the j-th row have the same colour. (The indices of the cells in a row are considered modulo 2^n.) Prove that the maximal possible number of colours is 2^n.

15. In a chess tournament $2n + 3$ players take part. Every two play exactly one match. The schedule is such that no two matches are played at the same time, and each player, after taking part in a match, is free in at least n next (consecutive) matches. Prove that one of the players who play in the opening match will also play in the closing match.